Combinatorial Reliability Analysis of Multi-State Systems Under Epistemic Uncertainty.

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Abstract
Multi-state systems (MSSs) are often found in real-world applications where components or the system as a whole can exhibit multiple performance levels or states. This multi-state nature poses significant challenges for reliability evaluation. Multi-valued decision diagrams (MDDs) are effective for assessing the reliability of MSSs under the assumption that system parameters are deterministic. However, in many real-world scenarios, it is difficult to ascertain the precise values of such parameters due to epistemic uncertainty. This paper addresses MDD-based reliability analysis of MSSs by integrating both interval theory and fuzzy set theory to account for epistemic uncertainty. The proposed methods are applied to a high-speed train bogie system to verify their effectiveness, with the results showing that the proposed methods provide practical reliability assessments under uncertain conditions.
Keywords: Multi-state system, multi-valued decision diagram, Epistemic uncertainty, Interval theory, Fuzzy set theory.

1. Introduction

Multi-state systems (MSSs), where components or the system can operate at multiple performance levels, are prevalent in various domains (Xing and Amari, 2015). Several methods have been developed for the reliability analysis of MSSs, including extensions of binary-state reliability models (Ramirez et al., 2004; Shrestha and Xing, 2008), Markov-based approaches (Cafaro et al., 1986), universal generation functions (Levitin, 2004; Nahas and Nourelfath, 2021), Bayesian networks (Zhou et al., 2006), simulation-based approaches (Pourhassan et al., 2021), and multi-valued decision diagrams (MDDs) (Xing and Dai, 2009). MDDs, as extensions of binary decision diagrams (BDDs), offer reduced computational complexity for large-scale MSSs (Xing and Amari, 2015).

Recent advancements in MDD-based reliability analysis have focused on various applications, such as phased-mission systems (Li et al., 2018), cloud computing systems (Mo and Xing, 2021), and social networks (Zhang et al., 2020). Despite this progress, the impact of uncertainty, particularly epistemic uncertainty, has not been adequately explored in MDD-based studies.

Epistemic uncertainty, which stems from incomplete knowledge or insufficient data, differs from aleatory uncertainty that arises from inherent randomness (Hu et al., 2021; Sarazin et al., 2021). Techniques such as interval theory (Sankararaman et al., 2011) and fuzzy set theory (Zadeh, 1965) have been used to model epistemic uncertainty. This paper integrates interval theory and fuzzy set theory into the MDD framework for MSS reliability analysis under epistemic uncertainty, proposing interval-MDD and fuzzy-MDD methods to address this issue. A case study involving a high-speed train bogie system demonstrates the applicability of the proposed methods.

2. Preliminary Model of MDDs

An MDD is a directed acyclic graph used to model the reliability of MSSs. The system's state is represented by leaf nodes ('1' for operational, '0' for failure), while intermediate nodes represent system components with multiple performance states (Xing et al., 2015; Xing et al., 2019). The MDD models the system's behavior through an expression like (1):

$$f = (x = 1)fx = 1 + (x = 2)fx = 2 + \cdots + (x = n)fx = n = case(x, f1, f2, \dots, fn)f \ = \ (x \ = \ 1)f\_\{x = 1\} \ + \ (x \ = \ 2)f\_\{x = 2\} \ + \backslash cdots \ + \ (x \ = \ n)f\_\{x = n\} \ = \ case(x, f\_1, f\_2, \backslash dots, f\_n)\ f = (x = 1)fx = 1 + (x = 2)fx = 2 + \cdots + (x = n)fx = n = case(x, f1, f2, \dots, fn)$$

where fxf_xfx represents the system's state for component xxx. Logical operations (AND, OR) combine sub-MDDs to form a complete system MDD (Xing et al., 2009), as shown in (2):

$$f \circ h = case(x, f1 \circ h1, \cdots, fn \circ hn) if\ index(x) = index(y) f \ \backslash circ\ h = \ case(x, f\_1 \ \backslash circ\ h\_1, \backslash cdots, f\_n \ \backslash circ\ h\_n) \ \backslash quad \ \backslash text\{if\ index\}(x) = \backslash text\{index\}(y) f \circ h = case(x, f1 \circ h1, \cdots, fn \circ hn) if\ index(x) = index(y)$$

Probabilities for each system state SkS_kSk are evaluated by recursively summing over all possible paths from the root to leaf node '1' (3):

$$Pk(f) = px,1(t)Pk(f1) + \cdots + px,n(t)Pk(fn)P\_k(f)$$
$$= p\_\{x,1\}(t)P\_k(f\_1) + \backslash cdots + p\_\{x,n\}(t)P\_k(f\_n)Pk(f)$$
$$= px,1(t)Pk(f1) + \cdots + px,n(t)Pk(fn)$$

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3. MDD-Based Uncertainty Reliability Analysis

3.1 Interval-MDD Method

When uncertainty exists in the probability values of system components, intervals can be used to represent these probabilities. Interval operations are defined as follows (4)-(8):

$$[a] \circ [b] = \{a \circ b \mid a \in [a], b \in [b]\}[a] \circ [b] = \{a \circ b \mid a \in [a], b \in [b]\}[a] \circ [b] = \{a \circ b \mid a \in [a], b \in [b]\}$$

$$For\ example, addition\ of\ intervals\ is\ represented\ as:$$

$$[a] + [b] = [a + b, a + b][a] + [b] = [a + b, a + b][a] + [b] = [a + b, a + b]$$

In the presence of epistemic uncertainty, the interval probabilities of each component are denoted as $[px,i](t)[p\_\{x,i\}](t)[px,i](t)$, and system state probabilities are calculated accordingly.

3.2 Fuzzy-MDD Method

Fuzzy numbers, representing vagueness, are employed to model uncertainty. Triangular fuzzy numbers are commonly used, with their membership function defined in (10):

$A(x) = \{x - ab - a, a \le x \le bc - xc - b, b \le x \le c0, otherwise A(x) = \llbracket \{cases\} \backslash frac\{x - a\}\{b - a\}, \& a \le x \le b \backslash\backslash \backslash frac\{c - x\}\{c - b\}, \& b \le x \le c \backslash\backslash 0, \& \backslash text\{otherwise\} \rrbracket \{cases\} A(x) = \{\{ \{b - ax - a, c - bc - x, 0, a \le x \le bb \le x \le c otherwise$

Fuzzy arithmetic operations, such as addition and multiplication, are applied to calculate fuzzy system state probabilities.

4. Case Studies

A case study of a high-speed train bogie system is used to demonstrate the proposed methods. Major components of the bogie system and their states are defined in Table 1, while the five system states are outlined in Table 2.

MDDs for the system being in different states are shown in Figures 3-6, and the system state probabilities are calculated using both the interval-MDD and fuzzy-MDD methods. Tables 4-6 present the results for different mileage values, showing that the fuzzy-MDD method provides more precise reliability estimates compared to the interval-MDD method.

4. Case Studies

To demonstrate the effectiveness of the proposed methods for handling epistemic uncertainty in the reliability evaluation of multi-state systems (MSSs), a detailed case study of a **high-speed train bogie system** is presented. The bogie system consists of several critical components, each capable of exhibiting multiple performance states due to various degradation modes. The reliability of the bogie system, therefore, relies on the combined performance of these components. In this study, we apply both the interval-MDD and fuzzy-MDD methods to analyze the system's reliability across different operating conditions.

4.1 System Overview and Component States

The high-speed train bogie system includes critical components like axles, wheels, air springs, and traction motors, which are essential for the safe and efficient operation of the train. Each of these components can exist in multiple states, ranging from fully operational to failed, with some components having intermediate degradation states.

The state definitions for the key components in the bogie system are summarized in **Table 1**. For instance, the axle can either be operational or failed, while the wheel has four distinct states: operational, minor abrasion, severe abrasion with unmet standards, and severe abrasion meeting all standards. These multi-state components reflect the real-world degradation behavior of the bogie system during high-speed train operations.

Table 1: Major Components in the High-Speed Train Bogie System

Component	State 1	State 2	State 3	State 4
Axle	Operational	Failed	-	-
Vertical Shock Absorber	Operational	Failed	-	-
Wheel	Operational	Abrasion	Stripped, not meeting standards	Stripped, meeting all standards

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Air Spring	Operational	Failed	-	-
Traction Motor	Operational	Failed	-	-

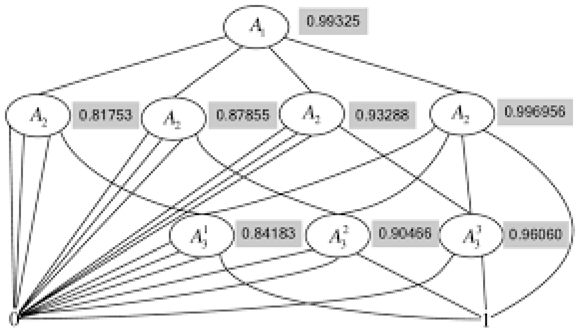
Table 2 defines the overall system states based on combinations of component states. The bogie system is considered to be in one of five states, ranging from full operational capacity to complete failure. For instance, in **State 3**, the system may have degraded wheels or failed air springs, but still functions in a reduced capacity.

Table 2: Definition of Bogie System States

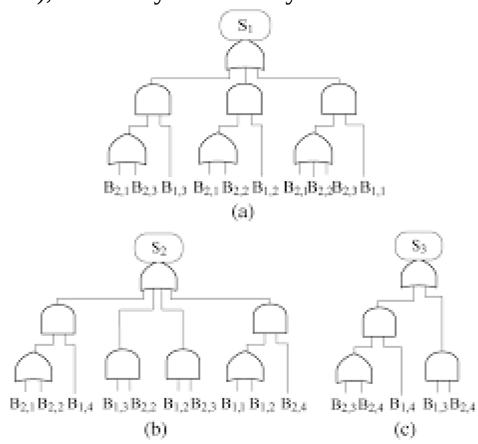
System State	**Condition**
State 1	All components are fully operational
State 2	Vertical shock absorber failed, other components operational
State 3	One or more components (wheel, air spring, etc.) in degraded condition
State 4	Severe degradation, multiple components degraded
State 5	Complete failure, any critical components failed

4.2 MDD Modeling of System States

To model the reliability of the bogie system, **Multi-Valued Decision Diagrams (MDDs)** are generated for each of the system states. The MDDs capture the relationships between the various component states and system-level outcomes. Figures 3 through 6 show the MDDs for states 1 to 4, respectively. State 5, representing complete system failure, is implicitly evaluated as the complement of the probabilities for states 1 to 4.



- **Figure 3** illustrates the MDD for **State 1**, where all components are operational. In this case, each component is in its best possible state (operational), and the system is fully functional.



- **Figure 4** shows the MDD for **State 2**, where the vertical shock absorber has failed, but all other components remain operational.

The MDD approach allows for efficient and accurate computation of the system's reliability by systematically combining the probabilities of individual component states using logical operations (AND, OR).

4.3 Reliability Evaluation Using Interval-MDD

The interval-MDD method is applied to evaluate the reliability of the bogie system under epistemic uncertainty. The interval approach models uncertainty by representing the failure probabilities of components as intervals, rather than fixed values. This approach captures the range of possible outcomes due to uncertainty in the underlying component reliability data.

The state probabilities for the bogie system, denoted as Pr(Si)Pr(S_i)Pr(Si), are calculated based on the MDDs for each system state (Equations 17–21). For instance, the probability of being in **State 1** is computed as the product of the operational probabilities for all components:

$$Pr(1) = Pr(A1) \times Pr(B1) \times Pr(C1) \times Pr(D1) \times \dots Pr(1)$$
$$= Pr(A\_1) \backslash times\ Pr(B\_1) \backslash times\ Pr(C\_1) \backslash times\ Pr(D\_1) \backslash times\ \backslash dotsPr(1)$$
$$= Pr(A1) \times Pr(B1) \times Pr(C1) \times Pr(D1) \times \dots$$

4.4 Reliability Evaluation Using Fuzzy-MDD

The fuzzy-MDD method extends the reliability analysis by incorporating fuzzy numbers to model epistemic uncertainty. In this approach, component failure probabilities are represented by triangular fuzzy numbers, which account for vagueness in the underlying data. The membership functions for the fuzzy numbers are defined as per Equation (10), with the midpoint representing the most likely value and the bounds representing the range of uncertainty.

The system state probabilities are recalculated using the fuzzy-MDD method, following the same procedure as for the interval-MDD. The results, shown in **Table 6**, indicate that the fuzzy-MDD method provides more detailed information by capturing the full range of possible reliability outcomes, from the lower to the upper bounds of the fuzzy numbers. This added granularity allows decision-makers to better understand the uncertainty associated with the system's reliability performance.

4.5 Comparison of Interval-MDD and Fuzzy-MDD Methods

The results of the interval-MDD and fuzzy-MDD analyses reveal that both methods effectively account for epistemic uncertainty in the reliability evaluation of the bogie system. However, the fuzzy-MDD method offers more precise reliability estimates, as it captures not only the range of possible outcomes but also the most likely reliability values through the use of fuzzy numbers.

4.6 Insights and Practical Implications

The case study demonstrates that the proposed interval-MDD and fuzzy-MDD methods are effective tools for evaluating the reliability of MSSs under uncertainty. In the context of high-speed rail systems, where safety and reliability are paramount, these methods provide valuable insights into the degradation behavior of key components like the axle, wheels, and traction motor.

By incorporating epistemic uncertainty into the analysis, railway operators can make more informed maintenance and operational decisions. For example, the results suggest that as the mileage of a high-speed train bogie system increases, the probability of system failure or degradation becomes more significant. This information can be used to schedule preventive maintenance activities before critical failures occur, ensuring both safety and cost-efficiency.

5. Conclusion and Future Work

This paper presented MDD-based methods for MSS reliability analysis under epistemic uncertainty, proposing both interval-MDD and fuzzy-MDD approaches. A high-speed train bogie system was analyzed to validate the methods. The fuzzy-MDD method was found to provide more accurate reliability estimates than the interval-MDD method. Future work will extend these methods to more complex systems and practical data sets.

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